Math 210

Fall 2009

Instructor : F. Brock

Exam

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1. (10 pts.) A function f is defined by $f(x) = x^3 \sin(\frac{1}{x^2})$, for $x \neq 0$, and $f(0) = 0$. Prove that f is differentiable and uniformly continuous on R.

2. (10 pts.) A sequence a_n is defined by $a_{n+1} = a_n + \frac{1}{a_n^2}$, and $a_1 = 1$. Obtain the asymptotic behavior of a_n, and prove your answer.

3. (10 pts.) Let $f : K \to R$ be a continuous function, where K is a closed and bounded interval. For $a \in R$ put $E = \{x \in K : f(x) \geq a\}$. Prove that E is a compact subset of R.

4. (10 pts.) Let $f : R \to R$ be a differentiable function, and assume that $|f'(x)| \leq A < 1$ for all $x \in R$. A sequence x_n is defined by $x_{n+1} = f(f(x_n))$. Prove that x_n is a convergent sequence irrespective of the value of x_1.

5. (10 pts.) Let f be given by $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$. Prove that f is defined and continuous on $-1 \leq x \leq 1$. Prove also that f is differentiable on $(-1, 1)$ and find its derivative.

6. (10 pts.) Consider the infinite series

$$\sum_{n=1}^{\infty} [nxe^{-nx^2} - (n-1)xe^{-(n-1)x^2}].$$

If S_n is the n^{th} partial sum, compute S_n, $\lim_{n\to\infty} S_n$, and determine whether or not the series converges uniformly in a nbhd $(-\delta, \delta)$ of 0.

7. (10 pts.) Evaluate the following limits and prove your answers:

$$(a)\ \lim_{n\to\infty} \int_0^n (1 + \frac{x}{n})^n e^{-2x} dx, \quad (b)\ \lim_{n\to\infty} \int_0^1 \frac{nt^{n+2}}{1+t} dt.$$

8.(10 pts.) State without proof four " deeper" properties of continuous functions.Prove that if f is continuous at a, and $f(a) \neq 0$, then $1/f$ is continuous at a.

9. (10 pts.) Define

$$f(x) = \int_x^{x+1} \sin(t^3) dt.$$

Show that

$$f(x) = \frac{1}{3}\{\frac{\cos(x^3)}{x^2} - \frac{\cos[(x+1)^3]}{(x+1)^2}\} - \frac{2}{9}\int_{x^3}^{(x+1)^3} \frac{\cos u}{u^{5/3}} du,$$

and conclude that $|f(x)| < C/x^2$, where C is a constant; and that $\int_0^{\infty} \sin(t^3) dt$ converges.

10. (10 pts.) If $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$, prove that $f'(x) = \sum_{n=1}^{\infty} \frac{-\sin nx}{n}$, for all $x \in (0, 2\pi)$.

11. (10 pts.) Prove that $\int_0^{\infty} \frac{\sin x}{\sqrt{1+x^2}} dx$ is a convergent integral, which is not absolutely convergent.

12. (15 pts.) Let ϕ and S_0 be continuous on $I : a \leq x \leq b$. Define a sequence S_n of functions on I by

$$S_n(x) = \alpha + \int_a^x \phi(t) S_{n-1}(t) dt, n \geq 1, \alpha \text{ fixed in } I.$$

(a) If S_n converges uniformly to a function S on I, prove that S is differentiable and that $S'(x) = \phi(x)S(x)$. (b) Show that $\{S_n\}$ is uniformly convergent on I.